On one Application of Newton’s Method to Stability Problem
Şerife Yılmaz
Department of Mathematics,
Faculty of Science, Anadolu University,
Eskisehir 26470, Turkey
serifeyilmaz@anadolu.edu.tr

Abstract— In this paper we consider stability problem for switched linear systems. This problem can be formulated as a convex minimization problem. By modifying the cost functions we apply the vector-valued Newton’s method.

Keywords— Switched system; Hurwitz stability; common quadratic Lyapunov function; Newton’s method

I. INTRODUCTION
Let $A$ be $n \times n$ real matrix. If all eigenvalues of $A$ lie in the open left half plane then $A$ is said to be Hurwitz stable. Hurwitz stability of $A$ is equivalent to the following: There exists positive definite symmetric matrix $P$ such that

$$ATP + PA < 0$$

where the symbol “$T$” stands for the transpose, and the symbol “$<$” for negative definiteness. Hurwitz stability of $A$ implies the asymptotic stability of the zero solution of the linear system

$$\dot{x} = Ax$$

where $x = x(t) \in \mathbb{R}^n$. If the matrix $A$ switches between $N$ matrices $A_1, A_2, ..., A_N$, i.e. $A \in \{A_1, A_2, ..., A_N\}$ then the obtained system

$$\dot{x} = Ax$$

is called a switched system. Sufficient condition for the asymptotic stability of the zero solution of (3) is the existence of quadratic Lyapunov function of the form

$$V(x) = x^TPx$$

where $P > 0$ and

$$A_i^TP + PA_i < 0 \quad (i = 1, 2, ..., N).$$

The matrix $P$ is called a common solution to the Lyapunov inequalities (4).

The stability problem of linear switched systems has been investigated in a lot of works (see [1-11] and references therein).

II. LYAPUNOV EQUATIONS
In this section, we consider the Lyapunov inequality (1) which is equivalent to the following equation.

$$A^TP + PA = -Q$$

where $Q > 0$. We are looking for a positive definite solution $P$ of (5). In the iteration steps, the obtained $P$ is guaranteed to be symmetric. The following theorem shows that in the case of Hurwitz stability of $A$ this implies the positive definiteness of $P$.

Theorem 1. Assume that $A$ is Hurwitz stable. If there exists a symmetric solution $P$ to (5) then $P > 0$.

Proof: Define

$$\bar{P} = \int_0^\infty e^{A^Tt}Qe^{At}dt$$

where $e^{At}$ stands for the matrix exponential.

Since $A$ is Hurwitz stable, the matrix $A^T$ is also Hurwitz stable. Therefore $e^{At}$ and $e^{A^Tt}$ define exponential functions with exponents $\text{Re}(\lambda_i) \cdot t < 0$ where $\lambda_i$ are the eigenvalues of $A$. This implies that the integral in (6) is well defined. The matrix $\bar{P}$ is symmetric, positive definite and satisfies the following relation

$$A^T\bar{P} + \bar{P}A = -Q$$

(see [15]). Then

$$A^T(P - \bar{P}) + (P - \bar{P})A = 0.$$

Multiplying by $e^{A^Tt}$ and $e^{At}$ give

$$0 = e^{A^Tt}[A^T(P - \bar{P}) + (P - \bar{P})A]e^{At}$$

$$= \frac{d}{dt}[e^{A^Tt}(P - \bar{P})e^{At}].$$
The integration from 0 to $\infty$ yields
$$[e^{At}(P - \tilde{P})e^{At}]_0^\infty = 0.$$  

Using the fact that $e^{At} \to 0$, $e^{At} \to 0$ as $t \to \infty$ we obtain
$$0 - (P - \tilde{P}) = 0$$
and $P = \tilde{P} > 0$.

We are looking for an iterative procedure for a common $P$ satisfying (4). Theorem 1 allows to guarantee positive definiteness of $P$ obtained at each step of iteration.

III. MODIFIED NEWTON’S METHOD

Consider a differentiable function $F: \mathbb{R}^n \to \mathbb{R}^n$ and the following equation

$$F(x) = 0.$$  

(7)

Here

$$x = (x_1, x_2, ..., x_r)^T \in \mathbb{R}^n,$$

$$F(x) = (f_1(x), f_2(x), ..., f_n(x))^T.$$ 

Denote the Jacobian matrix by $J(x)$, i.e.,

$$J(x) = \left( \frac{\partial f_i(x)}{\partial x_j} \right) (i, j = 1, 2, ..., n).$$

The Newton method is a method for an approximate solution of (7) and starting from a suitable initial point $x^0$, the iteration is defined by

$$x^k = x^{k-1} - J(x^{k-1})^{-1}F(x^{k-1}) \quad (k = 1, 2, ...).$$

Define

$$r = \frac{n(n+1)}{2}$$

and

$$P = P(x) = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_2 & x_{n+1} & \cdots & x_{2n-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_n & x_{2n-1} & \cdots & x_r \end{pmatrix}. $$

The matrix inequalities (4) are equivalent to

$$f_i(x) = \lambda_{\text{max}}(A^TP + PA_i) < 0 \quad (i = 1, 2, ..., N)$$

(8)

where $x = (x_1, x_2, ..., x_r)^T \in \mathbb{R}^r$, $\lambda_{\text{max}}(\cdot)$ stands for the maximal eigenvalue.

In the case of simple maximum eigenvalue of $A^TP(x) + P(x)A$, the gradient of $f(x) = \lambda_{\text{max}}(A^TP(x) + P(x)A)$ should be easily calculated. Indeed since the function $x \to A^TP(x) + P(x)A$ is linear then

$$A^TP(x) + P(x)A = \sum_{j=1}^{r} x_jQ_j.$$ 

Then $\nabla f(x) = (u^TQ_1u, ..., u^TQ_r, u)$, where “$\nabla$” stands for the gradient, $u$ is the unit eigenvector corresponding to the maximum eigenvalues of $A^TP(x) + P(x)A$ (see [12]).

**Proposition 1.** The function $f_i(x)$ is convex for each $i$.

**Proof:** The relation $P \to A^TP + PA_i$ is linear. On the other hand for symmetric $C$, the function $C \to \lambda_{\text{max}}(C)$ is convex [16]. Therefore $f_i(x)$ is convex as a composition of linear and convex functions.

□

The system (4) has a common solution $P > 0$ if and only if there exists $x, \in \mathbb{R}^r$ such that

$$f_i(x) < 0 \quad (i = 1, 2, ..., N).$$

(9)

In order to apply Newton’s method instead of the minimization of the functions $f_i(x)$, we consider the system of equations

$$f_i(x) = 0 \quad (i = 1, 2, ..., N).$$

Without loss of generality we can set $r = N$. Indeed if $N > r$, we can combine some function by using the operation maximum. For example if $N = r + 1$ then define

$$g_1(x) = \max(f_1(x), f_2(x)),$$

$$g_i(x) = f_{i+1}(x) \quad (i = 2, 3, ..., N).$$

This operation preserves convexity. If $N < r$ we use the operation of duplication. Thus from the now we assume that $r = N$.

Define $F = (f_1, f_2, ..., f_r)^T$ and consider the equation

$$F(x) = 0$$

(10)

where $x \in \mathbb{R}^r$.

If we apply the classical Newton’s method to (10) we obtain the trivial sequence $P_k \to 0$, since the functions $f_i(x)$ are positive homogenous. To avoid this we impose the condition $\text{trace}(P) = 1$. The following proposition shows that this does not violate the generality.

**Proposition 2.** Assume that $P > 0$ and $A^TP + PA < 0$. Then $A^TP + PA < 0$ where $P_{r1} = \frac{1}{\text{trace}(P)} P_r$.

**Proof:** From $P = (p_{ij}) > 0$ it follows that for all $x \in \mathbb{R}^n$, $x \neq 0$, $x^TPx > 0$. Taking

$$x^i = (0, ..., 0, 1, 0, ..., 0)^T,$$

we obtain $p_{11} > 0$. Therefore $\text{trace}(P) > 0$ and

$$A^TP_{r1} + PA = \frac{1}{\text{trace}(P)} [A^TP + PA] < 0.$$

The condition $\text{trace}(P) = 1$ reduces the number of variables from $r$ to $r - 1$. To solve (10) the following algorithm is suggested.

**Algorithm 1.**

1) Consider the equation (10). Take initial matrix $Q = \text{diag}(1, 2, 2, ..., 2)$ and consider $A^TP + PA_1 = -Q$.  

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The solution of this matrix equation let be $P_0$. Dividing $P_0$ by trace($P_0$) gives the initial iteration $x^0$.

2) Replace the functions $f_i(x)$ by $\tilde{f}_i(x) + \frac{1}{\text{trace}(P_0)} \times (i = 1, 2, ..., r - 1).$ Apply Newton’s iteration

\[ x^k = x^{k-1} - J(x^{k-1})^{-1} F(x^{k-1}) \]

where $F = (\tilde{f}_1, \tilde{f}_2, ..., \tilde{f}_{r-1}).$

3) If $\tilde{f}_i(x^k) < 0 (i = 1, 2, ..., r - 1)$ for some $k$ then stop. Otherwise continue.

**Example 1.** Consider the Hurwitz stable matrices

\[ A_1 = \begin{pmatrix} -1 & -4 \\ -1 & -8 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} -3 & 5 \\ -2 & 1 \end{pmatrix}. \]

The corresponding functions are:

\[ f_1(x) = \lambda_{\text{max}}(A_1^T P(x) + P(x) A_1), \quad f_2(x) = \lambda_{\text{max}}(A_2^T P(x) + P(x) A_2) \]

where

\[ P(x) = \begin{pmatrix} x_1 & x_2 \\ x_2 & 1 - x_1 \end{pmatrix}. \]

For the matrix

\[ Q = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \]

the unique solutions of

\[ A_1^T P + P A_1 = -Q \quad (i = 1, 2) \]

is

\[ P_0 = \begin{pmatrix} 0.972 & -0.472 \\ -0.472 & 0.361 \end{pmatrix}. \]

Hence

\[ \frac{1}{\text{trace}(P_0)} \cdot P_0 = \begin{pmatrix} 0.729 & -0.354 \\ -0.354 & 0.271 \end{pmatrix} \]

and take the initial point $x^0 = (0.729, -0.354)^T$. For this point calculations give the following maximum eigenvalues and its corresponding unit eigenvectors:

\[ \lambda_{\text{max}}(A_1^T P(x^0) + P(x^0) A_1) = -0.75 \]

the maximum eigenvector: $u^1 = (1,0)^T$,

\[ \lambda_{\text{max}}(A_2^T P(x^0) + P(x^0) A_2) = 0.8333 \]

the maximum eigenvector: $u^2 = (-0.7090, -0.7051)^T$.

Therefore

\[ f_1(x^0) = -0.75, \quad f_2(x^0) = 0.833. \]

and

\[ \nabla f_1(x)|_{x=x^0} = (-2, -2), \quad \nabla f_2(x)|_{x=x^0} = (2.988, 0.961). \]

Therefore the Jacobian matrix of $F(x) = (f_1(x), f_2(x))^T$ at $x^0$ is

\[ J(x^0) = \begin{pmatrix} -2 & -2 \\ 2.988 & 0.961 \end{pmatrix} \]

and

\[ x^1 = \begin{pmatrix} 0.729 \\ -0.354 \end{pmatrix} + (0.237, -0.737, -0.493, 0.833, 0.375) \]

\[ = \begin{pmatrix} -0.130 \\ -0.119 \end{pmatrix}. \]

After 3 steps, we get $f_1(x^3) < 0$ and $f_2(x^3) < 0$ (see Table I). Hence for the matrix

\[ P = P(x^3) = \begin{pmatrix} 0.390 & -0.247 \\ -0.247 & 0.609 \end{pmatrix}, \]

$A_1^T P + P A_1 < 0 (i = 1, 2)$ are satisfied.

**Example 2.** Consider the Hurwitz stable matrices

\[ A_1 = \begin{pmatrix} -1 & 2 \\ -1 & -2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -2 & -3 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad A_3 = \begin{pmatrix} 1 & 2 \\ -2 & -2 \end{pmatrix}. \]

The corresponding functions are:

\[ f_1(x) = \max(\lambda_{\text{max}}(A_1^T P(x) + P(x) A_1), \lambda_{\text{max}}(A_2^T P(x) + P(x) A_2)), \]

\[ f_2(x) = \lambda_{\text{max}}(A_3^T P(x) + P(x) A_3). \]

where

\[ P(x) = \begin{pmatrix} x_1 & x_2 \\ x_2 & 1 - x_1 \end{pmatrix}. \]

For the matrix

\[ Q = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \]

the unique solutions of

\[ A_1^T P + P A_1 = -Q \quad (i = 1, 2) \]

is

\[ P_0 = \begin{pmatrix} 0.416 & 0.083 \\ 0.083 & 0.583 \end{pmatrix} \]

and trace($P_0$) = 1. Take the initial point $x^0 = (0.416, 0.083)^T$. For this point calculations give the following maximum eigenvalues and its corresponding unit eigenvectors:

\[ \lambda_{\text{max}}(A_1^T P(x^0) + P(x^0) A_1) = -1, \quad u^1 = (1,0)^T, \]

\[ \lambda_{\text{max}}(A_2^T P(x^0) + P(x^0) A_2) = 0.680, \quad u^2 = (-0.082, 0.996)^T, \]

\[ \lambda_{\text{max}}(A_3^T P(x^0) + P(x^0) A_3) = 0.567, \quad u^3 = (-0.987, 0.160)^T. \]
Therefore
\[ f_1(x^0) = \max(-2.105,0.074) = 0.680, \]
\[ f_2(x^0) = 0.567. \]
and
\[ \nabla f_1(x)|_{x=x^0} = (-1.191,-5.767), \]
\[ \nabla f_2(x)|_{x=x^0} = (0.786,-3.478). \]
The Jacobian matrix of \( F(x) = (f_1(x),f_2(x))^T \) at \( x^0 \) is
\[ J(x^0) = \begin{pmatrix} -1.191 & -5.767 \end{pmatrix}. \]
Therefore
\[ x^1 = \begin{pmatrix} 0.416 \end{pmatrix} + \begin{pmatrix} -0.400 \\ 0.083 \end{pmatrix} \begin{pmatrix} 0.680 + 0.5 \\ -0.137 \end{pmatrix} = \begin{pmatrix} 0.180 \\ 0.336 \end{pmatrix}. \]
After 16 steps, we get \( f_1(x^{16}) < 0 \) and \( f_2(x^{16}) < 0 \) (see Table II). Hence for the matrix
\[ P = P(x^{16}) = \begin{pmatrix} 0.461 & 0.318 \\ 0.318 & 0.539 \end{pmatrix}, \]
\[ A_i^T P + PA_i < 0 \ (i = 1,2,3) \) are satisfied.

Table II

<table>
<thead>
<tr>
<th>( k )</th>
<th>( x^k )</th>
<th>( f_1(x^k) )</th>
<th>( f_2(x^k) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( (0.180,0.336)^T )</td>
<td>1.035</td>
<td>0.224</td>
</tr>
<tr>
<td>2</td>
<td>( (0.581,2.025)^T )</td>
<td>8.502</td>
<td>6.641</td>
</tr>
<tr>
<td>3</td>
<td>( (0.553,0.313)^T )</td>
<td>0.108</td>
<td>-0.122</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>15</td>
<td>( (0.293,0.919)^T )</td>
<td>2.527</td>
<td>1.514</td>
</tr>
<tr>
<td>16</td>
<td>( (0.461,0.318)^T )</td>
<td>-0.063</td>
<td>-0.074</td>
</tr>
</tbody>
</table>

**Example 3.** Consider the Hurwitz stable matrices
\[ A_1 = \begin{pmatrix} -32 & 5 & 12 \\ -10 & 1 & -2 \\ -9 & 7 & -17 \end{pmatrix}, A_2 = \begin{pmatrix} -4 & 5 & 2 \\ -6 & -11 & 3 \\ 1 & 0 & -10 \end{pmatrix}, \]
\[ A_3 = \begin{pmatrix} -5 & -3 & 1 \\ 2 & -4 & 2 \\ 4 & 1 & -5 \end{pmatrix}, A_4 = \begin{pmatrix} -6 & 1 & -2 \\ 3 & -3 & 4 \\ 1 & -2 & -4 \end{pmatrix}, \]
and
\[ A_5 = \begin{pmatrix} -10 & -4 & -2 \\ -7 & -8 & 20 \\ 7 & -2 & -22 \end{pmatrix}. \]
The corresponding functions are:
\[ f_i(x) = \lambda_{\text{max}}(A_i^T P(x) + P(x)A_i) \]
where
\[ P(x) = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & 1 - x_1 - x_4 \end{pmatrix}. \]
For the matrix
\[ Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \]
the unique solutions of
\[ A_i^T P + PA_i = -Q \ (i = 1,2) \]
is
\[ P_0 = \begin{pmatrix} 0.0512 & -0.138 & 0.027 \\ -0.138 & 0.588 & -0.128 \\ 0.027 & -0.128 & 0.093 \end{pmatrix}. \]
Hence
\[ \frac{1}{\text{trace}(P_0)} \cdot P_0 = \begin{pmatrix} 0.069 & -0.188 & 0.037 \\ -0.188 & 0.803 & -0.174 \\ 0.037 & -0.174 & 0.126 \end{pmatrix} \]
and the initial point is
\[ x^0 = (0.069,-0.188,0.037,0.803,-0.174)^T. \]
For this point calculations give the following:
\[ F(x^0) = (-1.363,1.952,0.408,1.843,11.909)^T, \]
\[ J(x^0) = \begin{pmatrix} -64 & -20 & -17.999 & 0 & 0.357 \\ -8.937 & -9.033 & 4.011 & 1.273 & 0.057 \\ -5.916 & -4.210 & -4.211 & -0.128 & 0.677 \\ 0.047 & -3.062 & -6.494 & 5.374 & -2.547 \\ 21.888 & -8.146 & 15.925 & 35.579 & -11.025 \end{pmatrix} \]
and
\[ x^1 = \begin{pmatrix} -0.090 \\ 0.252 \\ 0.079 \\ 0.617 \end{pmatrix} \]
and \( x^1 = (-0.201,-0.138,-0.044,0.281,-0.028)^T \).
After 36 steps, we get
\[ x^{36} = (0.421,-0.138,-0.044,0.281,-0.028)^T \]
and
\[ F(x^{36}) = (-0.590,-0.360,-0.608,-0.447,-0.164)^T. \]
Hence
\[ A_i^T P + PA_i < 0 \ (i = 1,2,3,4,5) \] where
\[ P = P(x^{36}) = \begin{pmatrix} 0.421 & -0.138 & -0.044 \\ -0.138 & 0.281 & -0.028 \\ -0.044 & -0.028 & 0.297 \end{pmatrix}. \]

**REFERENCES**


