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ROOT FINDING PROBLEM -A MODIFIED APPROACH

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Introduction

Inverse interpolation technique was employed to get a simple root of a given equation by some researchers including Muller, L.I.G Chambers etc., in the past. Chambers employed a quadratic formula in his Paper[1[to get a root of an equation by successive approximation of y= f(x) by parabolas through (a,f(a)),(b,f(b)),(c,f(c)) with a particular strategy*. It was an observation that under certain conditions /assumptions the polynomial approximation is more effective.

A practical procedural strategy was developed to the existing procedure given by L.I.G. Chambers. This approach was found to be more time saving inducing confidence while getting iterates. This may be claimed an added advantage while choosing successive parabolas while approximating y=f(x) in determining the simple root. Some illustrative examples presented may prove the same.

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Select **(a, A), (b, B), (c, C)** be three points on the curve **y=f(x). Let { a, b, c }** be our first *triplet* with errors **α , β , γ** say.

The parabola passing through these three points whose axis is parallel to the X-axis can be written as

x=a(y-B)(y-C)/(A-B)(A-C)+b(y-C)(y-A)/(B-C)(B-A) + c(y-A)(y-B)/(C-A)(C-B) (1)

The value of 'x', denoted by d, at which y vanishes is given by

d=a BC / (A-B) (A-C) + b CA / (B-C) (B-A) + c AB / (C-A) (C-B) (2)

*……………….

Assume that 'ε' is a root of the equation f(x) = 0 and assume that a=$\varepsilon+\alpha$, b=ε+β ,c=ε+γ are approximations to the required root. Let d=ε + δ,

We have

ε+δ=($\varepsilon+\alpha$)BC/(A-B)(A-C)+(ε+β)AC/(B-C)(B-A)+ (ε+γ)AB/(C-A)(C-B) (3)

It is clear from equation (3), we have

1=BC/(A-B)(A-C)+AC/(B-C)(B-A)+AB/(C-A)(C-B) (4)

and

δ = α BC / (A-B) (A-C) + β AC / (B-C) (B-A) + γ AB / (C-A) (C-B)

= α β γ (P-Q+P^2) (1+Terms of higher order) (5)

Where **P =f^{11} (ε)/2f^1 (ε), Q = f^{111} (ε) / 6 f^1 (ε)**

δ = k α β γ, where k is a constant. **(I^{**})**

Many Iterative formulae have been obtained for the solution of the equation f(x) = 0 and these are readily available. It is the purpose of this note to Indicate certain new Iterative-formulae which can be derived using equation (5)

The associated order of Convergence will be given by 'δ'. Obviously the method will not be applicable for a double root($f^1(\varepsilon)$ is not equal to zero) as in this case , the associated curve touches the X- axis and lies only on one side of it and cannot be approximated by a representation of as a Polynomial in y . For the same reason, it is clear that it is necessary for the root to be isolated.

From Here on wards we have investigated for a condition of Convergence. The work is forwarded under the Assumption that f(x) has no double root.

In equation (2) substitute the values α , β, γ and δ

ε+δ = ($\varepsilon+\alpha$)BC/(A-B)(A-C)+(ε+β)AC/(B-C)(B-A) + (ε+γ)AB/(C-A)(C-B)

δ = α BC/(A-B)(A-C)+βAC/(B-C)(B-A) + γ AB/(C-A)(C-B) (5)

δ = -[α BC(B-C) -βAC(A-C) + γ AB(A-B)] / (A-B)(A-C)(B-C)

ε-d = [k_1 (a-ε) –k_2 (b-ε) +k_3 (c-ε)] (6)

Where **k_1 = BC/(A-B)(A-C) ; k_2 =CA/(A-B)(B-C) ; k_3 =AB/(A-C)(B-C)**

respectively,

Operate function 'f' either sides on (6)

f (ε-d) = -k_1 f(ε-a) + k_2 f(ε-b)) -k_3 f(ε-c)

Apply Taylor's series and expand about at ε

f(ε)–d $f^1(\varepsilon)$+d^2/2! f^{11} (ε)–d^3/ 3! f^{111} (ε)+ ….= -k_1 [f(ε) –a $f^1(\varepsilon)$ +a^2/2! f^{11} (ε) –a^3/3! f^{111} (ε) + …] + k_2[f(ε) –b $f^1(\varepsilon)$ + b^2/2! f^{11} (ε) –b^3/3! f^{111} (ε)+….] - k_3 [f(ε) –c $f^1(\varepsilon)$ + c^2/2! f^{11} (ε) –c^3/3! f^{111} (ε) + ……]

Since 'ε' is a root of f(x) = 0 i.e. f(ε) =0

$$\Rightarrow -d f^1(\varepsilon) + d^2/2!\, f^{11}(\varepsilon) - d^3/3!\, f^{111}(\varepsilon) + \ldots = -k_1[-a f^1(\varepsilon) + a^2/2!\, f^{11}(\varepsilon) - a^3/3!\, f^{111}(\varepsilon) + \ldots] + k_2[-b f^1(\varepsilon) + b^2/2!\, f^{11}(\varepsilon) - b^3/3!\, f^{111}(\varepsilon) + \ldots] - k_3[-c f^1(\varepsilon) + c^2/2!\, f^{11}(\varepsilon) - c^3/3!\, f^{111}(\varepsilon) + \ldots]$$

$$= (k_1 a - k_2 b + k_3 c)\, f^1(\varepsilon)/1! + (-k_1 a^2 + k_2 b^2 - k_3 c^2)\, f^{11}(\varepsilon)/2! + (-k_1 a^3 - k_2 b^3 + k_3 c^3)\, f^{111}(\varepsilon)/3! + \ldots \quad (7)$$

Compare the co-efficient of $f^1(\varepsilon)$, $f^{11}(\varepsilon)$, $f^{111}(\varepsilon)$

$$k_1 a - k_2 b + k_3 c = -d \quad (8)$$

$$-k_1 a^2 + k_2 b^2 - k_3 c^2 = d^2 \quad (9)$$

$$k_1 a^3 - k_2 b^3 + k_3 c^3 = -d^3 \quad (10)$$

From equation (7)

$$-d f^1(\varepsilon)\,[1 - d/2!\, f^{11}(\varepsilon)/ f^1(\varepsilon) + d^2/3!\, f^{111}(\varepsilon)/ f^1(\varepsilon) + \ldots] = (k_1 a - k_2 b + k_3 c)\, f^1(\varepsilon)\,[1 + f^{11}(\varepsilon)/2!\, f^1(\varepsilon)\,(-k_1 a^2 + k_2 b^2 - k_3 c^2 / (k_1 a - k_2 b + k_3 c) + f^{111}(\varepsilon)/ 3!\, f^1(\varepsilon)\,((k_1 a^3 - k_2 b^3 + k_3 c^3) / (k_1 a - k_2 b + k_3 c)) + \ldots] \quad (11)$$

From (8), (9) & (10)

$$d^2 = -k_1 a^2 - k_2 b^2 + k_3 c^2$$

$$k_1 a^3 - k_2 b^3 + k_3 c^3 / -k_1 a^2 + k_2 b^2 + k_3 c^2 = -k_1 a^2 + k_2 b^2 - k_3 c^2 / k_1 a - k_2 b + k_3 c$$

From equation (11) we can compare the like powers

i.e. $f^{11}(\varepsilon)/2!\, f^1(\varepsilon) = f^{111}(\varepsilon)/ 3!\, f^1(\varepsilon)$

»» $3 f^{11}(\varepsilon) = = f^{111}(\varepsilon)$ **(**)**

(**) is the condition for convergence of the root 'ε' to the Exact root

Finally from equation **(I)** $\boldsymbol{\delta = k\alpha\beta\gamma}$ approaches to zero as the data points (TRIPPLET) are close.

Where k is defined as $\mathbf{k = P - Q + P^2}$ **(12)**

Here k is called correction factor which is take care of ' δ ' to approaches to zero. Once ' δ ' tends to zero then error is minimized.

EXAMPLES

Example 1: Find a simple real root of the equation

$\mathbf{f(x) = x^3 - 5x + 1 = 0}$ by Inverse Interpolation method

Select the data points **(1, -3), (2, -1)** and **(4, 45)** Here we are selecting the data points such that one pair of functional values must have opposite signs.

Apply Quadratic Interpolation technique on the selected Triplet, we have

$$\mathbf{y = g(x) = 7x^2 - 19x + 9} \quad (13)$$

Now apply Inverse Interpolation Technique on the selected Triplet the first approximation from equation (1) we have the first Approximation is

$\mathbf{x_1 = 2.46731305}$

Find the value of $\mathbf{f(x_1)}$

Here three cases shall arise. They are

1) $f(x_1) > 0$ or 2) $f(x_1) < 0$ or 3) $f(x_1) = 0$. In the third case declare x_1 is the root. Otherwise select the suitable triplet by the method of Intervel Halfing (Modified) method. Continue this process till the desired level of accuracy.

If we adopt this process the sequence of approximations are

$\mathbf{x_1 = 2.46731305}$ **,** $\mathbf{x_2 = 2.085921302}$ **,**

$\mathbf{x_3 = 2.123650295}$**,** $\mathbf{x_4 = 2.102331889}$

$\mathbf{x_5 = 2.11756703}$ **,** $\mathbf{x_6 = 2.127819076}$ **,**

$\mathbf{x_7 = 2.128419060}$

the seventh Iteration value of 'x ' equation **(I)** gives satisfactory value of δ which is tends to zero.

Example 2: $\mathbf{x^3 - 15x + 4 = 0}$

By applying the similar procedure by selecting the three data points (Triplet)

(0.1, 3.986), (0.25, 0.265625), (0.5, -3.375) the Iterative values are as follows.

$\mathbf{x_1 = 0.26478752}$ **,** $\mathbf{x_2 = 0.267950209}$ **,**

$\mathbf{x_3 = 0.267949182}$ **,** $\mathbf{x_4 = 0.267949190}$

Test condition: At the real simple root the value $\boldsymbol{\delta = k\alpha\beta\gamma}$ *approaches to zero.*

MODIFIED BISECTION METHOD

As we know that The familiar Bisection method to find a real root of an equation f(x) =0. Here we are selecting an interval such that in that interval the function has opposite signs say f(a)f(b) < 0. There is a root lies between (a, b). Then we know that the first approximation $x_1 = (a + b)/2$. Then again check the condition for opposite signs and select a new interval by the above formula. This method will give accurate value i.e. convergence is definite but the process takes enormous computational work.

Iteration number	Bisection method	Modified Bisection method
1	0.5	0.492307692
2	0.75	0.723543123
3	0.875	0.811079116
4	0.8125	0.767311119
5	0.787125	0.745427121
6	0.76856	0.75636912
7	0.75928	0.75089812
8	0.76392	0.754677666
9	0.7616	0.75477766
10	0.76508	0.754778664

We are proposed a new method by generating the risky – parameters c_1, c_2, c_3, c_4, & **c** where the sequence of values are 0.496124031, 0.492307692, 0.484848484, 0.470588235, & 0.5 respectively. By selecting this sequence of parameters the method converges to the exact root in less number of iterations. For clarification we have solved problems on it.

Example 1: Find a real root of the equation $x^3 + x^2 - 1 = 0$. The root lies between (0, 1). If we apply the Interval Halfling method (Bisection method) we have the following approximations. $x_1 = 0.5$, $x_2 = 0.75$, $x_3 = 0.875$, $x_4 = 0.8125$, $x_5 = 0.787125$, $x_6 = 0.76856$, $x_7 = 0.75928$, $x_8 = 0.76392$, $x_9 = 0.7616$, $x_{10} = 0.76508$. The approximate root of the given equation is 0.754877666.

Modified Bisection method: Here the formula similar to that of the Bisection method with slight change .Here $x_{new} = c_K(x_R + x_L)$ where k can takes the values 1,2,3,4 and finally c_K stops at 0.5

$x_1 = 0.492307692$, $x_2 = 0.723542399$,

$x_3 = 0.811078775$, $x_4 = 0.767310587$,

$x_5 = 0.749524099$, $x_6 = 0.762514949$,

$x_7 = 0.762514949$, $x_8 = 0.756019524$.

It is very close to the root in eighth iteration where as the Bisection method not matched upto two significant figures. where as in this new technique we can observe the fact that it is matched upto two significant figures in less iterations.

Example 2 : $f(x) = x^3 - 15x + 4 = 0$ If we apply Bisection method , Modified Bisection method the results are as follows.{ The root lies between 0 and 1 }

The actual Root : 0.267949192

Iteration Number	Bisection method	The actual Root : 0.267949192	Modified Bisection Method
1	0.5	0.267949192	0.492307692
2	0.25	0.267949192	0.238694638
3	0.375	0.267949192	0. 344001096
4	0.3125	0.267949192	0.291347867
5	0.28185	0.267949192	0.267021248
6	0.2734375	0.267949192	0.266857938
7	0.26953125	0.267949192	0.267039593
8	0.267578125	0.267949192	0.267444392
9	0.268554687	0.267949192	0.267696792
10	0.268066406	0.267949192	0.267822992
11	0.267822265		
12	0.267944335		
13	0.2678833		
14	0.267913817		

Remarks: Bisection method is modified with the introduction of *"**Risk factor**"*

The convergence of Iterates is observed to be fairly good

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