On New Techniques for Optimization over Polynomials

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***Abstract*— In this paper two methods for optimization over polynomials are presented: Bernstein expansion and linear matrix inequalities approach. We mainly focus on the Bernstein expansion.**

***Keywords*— Bernstein expansion; stability; multivariable polynomial; linear matrix inequality**

I. Introduction

We give some examples from the control theory which are relevant to polynomial optimization.

i) Consider the following nonlinear system ([1])

$$\begin{aligned} \dot{x} &= f(x) \\ f(0) &= 0 \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $x = x(t)$, t is the time. Global asymptotic stability of the equilibrium solution $x(t) \equiv 0$ can be investigated by looking for a Lyapunov function $v: \mathbb{R}^n \to \mathbb{R}$ such that

$$v(0) = 0,$$

$$v(x) > 0,$$

$$< \nabla v(x), f(x) > < 0 \quad (\forall x \neq 0)$$

and $v(x) \to \infty$ as $\|x\| \to \infty$. Here $<\cdot,\cdot>$ stands for the scalar product. If f and v are polynomial the problem reduces to polynomial optimization.

ii) Let $A = \left.\frac{\partial f}{\partial x}\right|_{x=0}$ be the Jacobian matrix of f at $x = 0$. Then in (1) the origin is globally asymptotically stable if there exists a matrix P such that

$$P > 0, A^T P + PA < 0 \tag{2}$$

that is the matrix A is Hurwitz stable (the eigenvalues lie in the open left-half plane), where matrix inequalities in (2) means positive and negative definiteness. After parametrization of P, condition (2) can be reduced to the feasibility problem of polynomial inequalities.

iii) Consider third order uncertain polynomial

$$s^3 + a_1(q)s^2 + a_2(q)s + a_3(q) \tag{3}$$

where $q \in Q$ is an uncertainty vector from a box Q, the functions $a_i(q)$ are polynomially dependent on q. Is the family (3) robustly stable, i.e. all roots lie in the open left-half plane? Well known stability conditions ([2]) give

$$a_1(q) > 0, a_2(q) > 0, a_3(q) > 0,$$

$$a_1(q)a_2(q) - a_3(q) > 0$$

for all $q \in Q$. The problem is reduced to positivity of four polynomials over Q.

iv) For the linear control system $\dot{x} = Ax + Bu$ the feedback $u = Kx$ gives the closed loop system $\dot{x} = (A + BK)x$. Is there a feedback K such that the obtained system is globally asymptotically stable? Treating the entries of K as an uncertainty parameter q, the problem can be reduced to the following: Is there $q = (q_1, q_2, \dots, q_l)^T$ such that the matrix

$$A_0 + a_1 A_1 + \cdots + q_l A_l \tag{4}$$

is stable? In the case of symmetric matrices $A_0, A_1, \dots, A_l$ stability is equivalent to negative definiteness. Therefore the above problem is reduced to the following: Is there q such that

$$A_0 + q_1 A_1 + \cdots + q_l A_l < 0\ ? \tag{5}$$

Leading principal minor condition gives n polynomial inequalities.

v) Matrix root clustering problem seeks a criterion for a matrix to have its eigenvalues in a prescribed subregion of the complex plane. Consider the following region

$$\Omega = \{z \in \mathbb{C}: \operatorname{Re} f_i(z) < 0,\ i = 1,2, \dots, m\}$$

where $f_i(z)$ are polynomials. A necessary and sufficient condition for A to have all its eigenvalues in Ω is that there exists $P > 0$ satisfying

$$\{f_i(A)\}^T P + P f_i(A) < 0 \quad (i = 1,2, \dots, m)$$

([3]). Parametrization of P gives a feasibility problem for polynomial inequalities.

II. Bernstein Expansion

Let $L = (i_1, i_2, \dots, i_m)$ be m-tuple of nonnegative integers and for $x = (x_1, x_2, \dots, x_m)$

$$x^L = x_1^{i_1} x_2^{i_2} \cdots x_m^{i_m}.$$

For $N = (n_1, n_2, \dots, n_m)$

$$L \leq N \iff 0 \leq i_k \leq n_k \quad (k = 1,2, \dots, m).$$

An m-variate polynomial $p(x)$ is defined as

$$p(x) = \sum_{L\leq N} a_L x^L \quad (x \in \mathbb{R}^m). \tag{6}$$

Here $d = n_1 + n_2 + \cdots + n_m$ is called the degree of the polynomial $p(x)$.

The Lth Bernstein polynomial of degree d is defined by

$$B_{N,L}(x) = b_{n_1,i_1}(x_1)\cdots b_{n_m,i_m}(x_m) \quad (x \in \mathbb{R}^m) \tag{7}$$

where $b_{n,i}(t) = \binom{n}{i} t^i (1-t)^{n-i}$. The transformation of a polynomial from its power form (6) into its Bernstein form result in

$$p(x) = \sum_{L\leq N} p_L(U) B_{N,L}(x), \tag{8}$$

where the Bernstein coefficients $p_L(U)$ of p over the m-dimensional unit box $U = [0,1]\times\cdots\times[0,1]$ are given by

$$p_L(U) = \sum_{J\leq N} \frac{\binom{L}{J}}{\binom{N}{J}} a_J \quad (L \leq N) \tag{9}$$

Here $\binom{N}{L}$ is defined as $\binom{n_1}{i_1}\cdots\binom{n_m}{i_m}$. In [4], a difference table method for computing the Bernstein coefficients efficiently that avoids the binomial coefficients and product appearing in (9) is described.

Denote

$$\underline{m} = \min\{p(x): x \in U\},$$
$$\overline{m} = \max\{p(x): x \in U\},$$
$$\alpha = \min\{p_L(U): L \leq N\},$$
$$\beta = \max\{p_L(U): L \leq N\}.$$

Theorem 1 ([5]). The inequalities

$$\alpha \leq \underline{m} \leq \overline{m} \leq \beta \tag{10}$$

are satisfied.

Theorem 1 gives the bounds for the range of (6) over the unit box U. In order to obtain the Bernstein coefficients and bounds over an arbitrary box D, the box D should be affinely mapped onto U. To obtain convergent bounds for the range of the polynomial (6) over the box U, the box U should be divided into two boxes. If the division is continued and one calculates the minimal and maximal Bernstein coefficients in each subdivision step, the calculated bounds converge to the exact bounds (provided that the diameter of subboxes tends to zero).

If $\alpha > 0$ ($\beta < 0$) then the polynomial is positive (negative) on U. If $\alpha \leq 0$, $\beta \geq 0$ then by the bisection in the chosen coordinate direction the box U is divided into two boxes. A new box on which the inequality $\alpha > 0$ or $\beta < 0$ is satisfied should be eliminated, since our polynomial has constant sign on this box. Otherwise the box should be divided into two new boxes.

If a multivariate polynomial is positive (negative) on the box D the algorithm gives an affirmative answer after a finite number of steps

Example 1. Consider the following matrix family

$$A(q) = \begin{pmatrix} q_1 + 0.2 & q_2 \\ q_3 & q_1 - q_2 \end{pmatrix},$$
$$q_1 \in [-0.3, 0.4],\ q_2 \in [0, 0.3],\ q_3 \in [-1, 0].$$

$A(0,0,0) = \begin{pmatrix} 0.2 & 0 \\ 0 & 0 \end{pmatrix}$ is Schur stable (all eigenvalues lie in the unit open disc). Then the family $A(q)$ is robust Schur stable if and only if the determinant function

$$\begin{aligned} f(t, q_1, q_2, q_3) &= \det(A(q)^2 - 2tA(q) + I) \\ &= t^2q_1^2 - t^2q_1q_2 - t^2q_2q_3 - 2tq_1^3 + 3tq_1^3 \\ &+ 3tq_1^2q_2 - tq_1q_2^2 + 2tq_1q_2q_3 - tq_2^2q_3 \\ &+ q_1^4 - 2q_1^3q_2 + q_1^2q_2^2 - 2q_1^2q_2q_3 \\ &+ 2q_1q_2^2q_3 + q_2^2q_3^2 + 0.2t^2q_1 - 0.2t^2q_2 \\ &- 0.6tq_1^2 + 0.8tq_1q_2 - 0.2tq_2^2 + 0.2tq_2q_3 \\ &+ 0.4q_1^3 - 0.8q_1^2q_2 + 0.4q_1q_2^2 - 0.4q_1q_2q_3 \\ &+ 0.4q_2^2q_3 - 2.04tq_1 + 1.04tq_2 + 2.04q_1^2 \\ &- 2.08q_1q_2 + 1.04q_2^2 + 2q_2q_3 - 0.2t \\ &+ 0.4q_1 + 1.04 \end{aligned}$$

is nonzero, where $t \in [-1,1]$ ([6]). After 15 bisection and eliminations, we decide that $f > 0$ and this family is robust Schur stable.

III. LINEAR MATRIX INEQUALITIES (LMI)

LMI is an expression of the form ([7])

$$F(x) = F_0 + x_1F_1 + \cdots + x_nF_n < 0 \tag{11}$$

or equivalently

$$\lambda_{\max}(F(x)) < 0.$$

Here x_i $(i = 1,2,\ldots,n)$ are scalars, F_i $(i = 0,1,\ldots,n)$ are symmetric matrices.

More generally, LMI is the inequality

$$F(x) < 0, \ x \in M \tag{12}$$

where $F: \mathcal{X} \to \mathcal{S}$, $\mathcal{X}$ is finite dimensional, M is an affine set, $\mathcal{S}$ is the space of real symmetric $n \times n$ matrices and F is an affine function.

The main advantages of this problem are:

1) Convexity
2) The gradient vector of the function $\lambda_{\max}(F(x))$ can be evaluated by involving eigenvectors of $F(x)$.

For example, if $\lambda_{\max}(F(x_*))$ is unique then

$$\nabla\lambda_{\max}(F(x_*)) = u^T \nabla F(x_*) u$$

where u is the unit eigenvector corresponding to $\lambda_{\max}$.

The solution algorithms of LMI are based mainly on interior point method for convex optimization.

A primary issue in control systems concerns positivity of polynomials and this issue can be investigated via LMI techniques.

An interesting way of establishing whether p is positive consists of establishing whether p is SOS, i.e. can be written as

$$p(x) = \sum_i p_i(x)^2.$$

If a polynomial $p(x)$ is the sum of monomials of the same degree then $p(x)$ is called homogenous. Any polynomial can be viewed as a homogenous polynomial with one more variable set to 1. Establishing whether a homogenous polynomial is SOS amount to solving a convex optimization problem. Indeed, any homogenous $p(x)$ of degree $2d$ can be represented via

$$p(x) = x^{\{d\}^T}(H + L(\alpha))x^{\{d\}}$$

where $\sigma = \binom{d+n-1}{d}$, $x \in \mathbb{R}^n$ and $x^{\{d\}} \in \mathbb{R}^\sigma$ is a power vector for the polynomials of degree d and is a vector containing a basis for such polynomials. For example, if $d = 2$, $x^{\{d\}} = (x_1^2, x_1x_2, x_2^2)^T$. Here H is any symmetric matrix satisfying

$$h(x) = x^{\{d\}^T} H x^{\{d\}},$$

$L(\alpha)$ is a linear parametrization of the linear space

$$\mathcal{L} = \left\{L = L^T:\ x^{\{d\}^T} L x^{\{d\}} = 0\right\}.$$

Theorem 2 ([8]). $p(x)$ is SOS if and only if there exists α such that

$$H + L(\alpha) \geq 0. \qquad (13)$$

The condition (13) is an LMI feasibility test.

Example 2. Consider SOS polynomial

$$p(x_1, x_2) = x_1^4 - 3x_1^2x_2^2 + 4x_2^2.$$

In this example

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \qquad L(\alpha) = \begin{pmatrix} 0 & 0 & \alpha \\ 0 & -2\alpha & 0 \\ \alpha & 0 & 0 \end{pmatrix}$$

and $H + L(\alpha)$ is positive definite for $\alpha = -7/4$.

IV. Positivity over the Simplex

There are many problems from linear system theory which can be reduced to positivity of a multivariate polynomial over the simplex set

$$S = \left\{x \in \mathbb{R}^n:\ \sum_{i=1}^{n} x_i = 1,\ x_i \geq 0\right\}.$$

For example, nonsingularity problem of a matrix polytope

$$\mathcal{A} = \mathrm{conv}\{A_1, A_2, \dots, A_{n+1}\}$$

is equivalent to

$$p(x) = p(x_1, x_2, \dots, x_n)$$
$$= \det(x_1A_1 + \dots + x_nA_n + (1 - x_1 - \dots - x_n)A_{n+1}) \neq 0$$

over the set S. Positivity (or negativity) of $p(x)$ over S can be tested by the following algorithm.

Algorithm 1.

1) Consider the unit box $B = \{(x_1, x_2, \dots, x_n):\ 0 \leq x_i \leq 1\ \ (i = 1,2, \dots, n)\}$ which is the minimal box containing the set S. Define the Bernstein coefficients over B. If the array of Bernstein coefficients is positive then stop. Otherwise go to 2).
2) Divide B in two subboxes in the chosen coordinat direction and repeat 1) for both subboxes.
3) If the Bernstein coefficients over a subbox $D = [\alpha_1, \beta_1] \times \dots \times [\alpha_n, \beta_n]$ are positive or $\alpha_1 + \dots + \alpha_n \geq 1$ then eliminate the subbox D (in the last case D remains outside S).

Example 3. Consider nonsingularity problem of the polytope $\mathcal{A} = \mathrm{conv}\{A_1, A_2, A_3\}$, where

$$A_1 = \begin{pmatrix} 0 & -2 & 3 \\ 3 & -4 & -3 \\ -1 & 1 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} -3 & -3 & -3 \\ 1 & -2 & 1 \\ -1 & -2 & -2 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} -2 & -3 & 3 \\ -1 & -3 & 0 \\ -1 & 1 & -1 \end{pmatrix}.$$

We have

$$\Lambda = \{(\lambda_1, \lambda_2): \lambda_1 \in [0,1], \lambda_2 \in [0,1], \lambda_1 + \lambda_2 \leq 1\},$$
$$\mathcal{A} = \{\lambda_1A_1 + \lambda_2A_2 + (1 - \lambda_1 - \lambda_2)A_3: (\lambda_1, \lambda_2) \in \Lambda\}$$

and the determinant function of this family is

$$f(\lambda_1, \lambda_2) = 6\lambda_1^3 + 11\lambda_1^2\lambda_2 - 85\lambda_1\lambda_2^2 - 34\lambda_2^3 - 24\lambda_1^2 + 45\lambda_1\lambda_2 + 65\lambda_2^2 + 12\lambda_1 - 37\lambda_2 + 15.$$

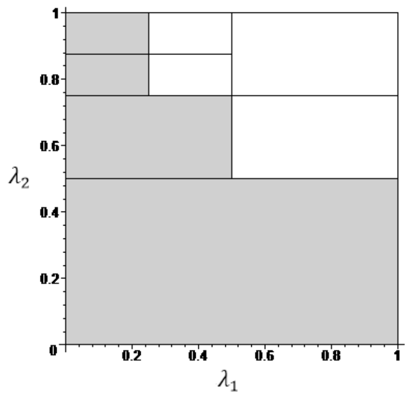

Figure 1. Bisection of rectangles.

The array of Bernstein coefficients (9) is

$$B(U) = \begin{pmatrix} 15 & 8/3 & 12 & 9 \\ 19 & 35/3 & 149/9 & -1/3 \\ 15 & 125/9 & 140/9 & -14 \\ 9 & 46/3 & 15 & -26 \end{pmatrix}$$

and has no constant sign. Therefore the bisection procedure must be applied to this problem. The algorithm reports after 0.187 s that the determinant function $f(\lambda_1, \lambda_2)$ is positive on the set Λ. It requires 8 bisection steps (Fig. 1). Note that f is negative for $\lambda_1 = 0.7$, $\lambda_2 = 0.8$.

LMI approach gives the following result for this problem ([8]):

A homogenous polynomial $p(x)$ is positive on $S \Leftrightarrow$ there exists natural number k such that

$$p(x)(x_1 + x_2 + \dots + x_n)^k > 0$$

for all $x \in \mathbb{R}^n$.

As we can see this condition increases the degree and contains an uncertain parameter k.

V. Feasibility of a System of Polynomial Inequalities

As pointed out in the Introduction there are many control problems which can be formulated as a feasibility problem of a system of polynomial inequalities.

Consider the following system of inequalities

$$p_i(x) > 0\ \ (i = 1,2, \dots, k),\ x \in Q \qquad (14)$$

where $p_i(x)$ are multivariate polynomials and Q is a box in $\mathbb{R}^n$. Is there $x_* \in Q$ such that $p_i(x_*) > 0$ for all $= 1,2, \dots, k$?

Using the fact that the Bernstein expansion and division procedure gives outer approximation of the range set (Theorem 1) and this approximate range tends to the exact range the following algorithm is suggested:

Algorithm 2.

1) Consider (14). Calculate the intervals which contain the images $p_i(Q)$ by using the Bernstein expansion.
2) If all left bounds are greater than zero then stop: $p_i(x) > 0$ for all $i = 1,2, \dots, k$ and $x \in Q$. If at least one upper bound is less than or equal to zero, then stop: the feasible set of (14) is empty. Otherwise apply the next step.
3) Divide the box Q into two subboxes in the chosen coordinate direction. For each subbox repeat 1) - 2).

4) Eliminate a subbox on which at least one upper bound is less or equal than zero.

The algorithm is finished if all subboxes are eliminated (the feasibility set is empty) or all left bounds are greater than zero (the subbox satisfies (14)).

Example 4.

$$p_1(x) = x_1^2 x_2 x_3 - 2x_2 x_3 + 6x_1 x_2 + 3x_1 x_3,$$
$$p_2(x) = x_1^2 x_2^2 x_3 - x_2^2 x_3 - x_1 x_2.$$

Is there $x \in [-3,3] \times [-3,3] \times [-3,3]$ such that $p_1(x) > 0$, $p_2(x) > 0$?

Figure 2 shows feasible set obtained by Algorithm 2.

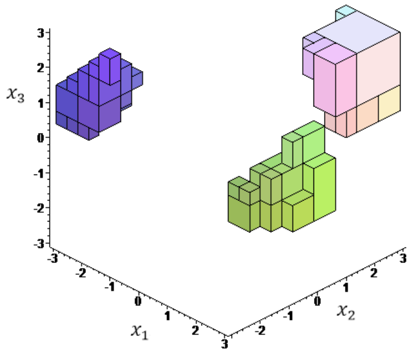

Figure 2. Feasible set.

LMI result for this problem is the following: Consider the set

$$\mathcal{X} = \{x \in \mathbb{R}^n : p_1(x) \geq 0, \ \dots, p_k(x) \geq 0\},$$

where $p_i(x)$ are polynomials. Then $\mathcal{X} = \emptyset \Leftrightarrow$ there exists $p \in C(p_1, \dots, p_k)$ such that $p(x) + 1 \equiv 0$ where $C(p_1, \dots, p_k)$ is the cone generated by polynomials $p_1, \dots, p_k$ (see [8]). We can see from this that the use of this result is not convenient.

VI. ROBUST STABILITY OF POLYNOMIAL MATRIX FAMILY

Consider a matrix family $A(q)$ where all entries depend polynomially on a scalar parameter $q \in [0,1]$. Is the family $\{A(q): q \in [0,1]\}$ robust stable, i.e. all matrices are stable? As proved in [9] the family $A(q)$ is robustly stable if and only if specially constructed two polynomials $f_1(q)$ and $f_2(q)$ are positive on $[0,1]$. LMI approach gives the following result ([9]): The problem can be reduced to LMI feasibility test, namely the family $A(q)$ is robust stable if and only if there exist numbers β_1, β_2, matrices Γ_1, Γ_2, vectors $\Delta_1 \in \mathbb{R}^{c_1}$, $\Delta_2 \in \mathbb{R}^{c_2}$ satisfying the following LMIs

$$\beta_1 > 0, \beta_2 > 0, \Gamma_1 \geq 0, \Gamma_2 \geq 0,$$
$$F_1 - \beta_1 R_1 - S_1(\Gamma_1) + T_1(\Delta_1) \geq 0,$$
$$F_2 - \beta_2 R_2 - S_2(\Gamma_2) + T_2(\Delta_2) \geq 0.$$

Here F_1, F_2, R_1, R_2, $S_1(\Gamma_1)$, $S_2(\Gamma_2)$, $T_1(\Delta_1)$, $T_2(\Delta_2)$ are symmetric matrices related to the polynomials f_1 and f_2. The total number of LMI scalar variables is given by

$$\eta = \sum_{i=1}^{2} \left(1 + \frac{m_i(m_i+1)}{2} + c_i\right)$$

where m_i is the degree of the polynomial f_i and $c_i = \frac{m_i(m_i-1)}{2}$. An example of 4×4 matrix polynomial with scalar $q \in [0,1]$ requires 210 parameters.

For the above mentioned example LMI relations give negative answer on robust stability. On the other hand the Bernstein expansion gives whole stability regions in $q \in [0,1]$.

Example 5.

Consider the same family from [9]

$$A(q) = \begin{pmatrix} 0 & 1 & 0 & 2-q \\ -1-q^2 & -2 & 7q-1 & 0 \\ -q^3 & 1-q & -1 & 0 \\ q & 0 & q^4 & -1 \end{pmatrix},$$

and $q \in [0,1]$. For this family

$$f_1(q) = -q^8 + q^7 + 3q^6 - 3q^5 + 16q^4 - 23q^3 + 20q^2 - 6q + 1,$$
$$f_2(q) = -q^{16} + 4q^{15} - 4q^{14} + 14q^{12} - 30q^{11} - 8q^{10} + 36q^9 - 75q^8 + 34q^7 + 35q^6 - 48q^5 + 170q^4 - 298q^3 + 440q^2 - 356q + 99.$$

Bernstein coefficients of $f_1(q)$ are:

$$1, \frac{1}{4}, \frac{3}{14}, \frac{27}{56}, \frac{61}{70}, \frac{11}{8}, \frac{31}{14}, \frac{33}{8}, 8$$

and we conclude that $f_1(q)$ is positive for all $q \in [0,1]$ by Theorem 1. For the polynomial $f_2(q)$, after 4 bisection steps we decide that $f_2(q) < 0$ for all $q \in \left[\frac{5}{8}, \frac{11}{16}\right] = [0.625, 0.6875]$. That is, the family $\{A(q): q \in [0,1]\}$ is not robust stable. Additionally, Bernstein expansion gives the following stability intervals:

$$\left[0, \frac{1}{2}\right] \cup \left[\frac{3}{4}, 1\right]$$

whereas above LMI relations can not give such regions.

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