On some Properties of Multi-Communication Dynamical System with Full Connection Structure

General Transport-Logistic Problem

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Abstract- A dynamical system with discrete time T is considered. There are N vertices. Each vertex sends messages to a vertex or $k \leq N$ successively numbered vertices at every time instant in accordance with a given rule during T. The *competition* takes place if any two vertices try to send messages to each other. Each vertex changes its state during T if there is no competition or the vertex wins the competition, and otherwise the vertex skips a time step. The average number of changes of a vertex state per a time unit is called the *vertex productivity.* The average productivity of the vertices is called the *system productivity.* The system is in the *state of synergy* if states of all vertices change after a time instant. Problems of system productivity, the existence of synergy and other qualitative properties depending on initial states and given rules are researched.

Keywords: transport-logistic problem; dynamical system; modeling; communication; synergy.

1. INTRODUCTION

Concepts of dynamical systems are used in physics, communication, transport models, etc., [1 – 11].

M. Kac has introduced a discrete dynamical system, which has the property of reversibility, [1]. This model was used in the analysis of the behavior on the basis of thermodynamics laws.

A new approach of physical phenomena research with using of the Kac model was introduced by V.V. Kozlov in [2].

A dynamical system, called the game "Life", has been considered in [3]. This system has been introduced by J.H. Conway. There is an infinite two- dimensional structure. Each cell of this structure can be in one of two states. The state of each cell can change in discrete instants depending on the states of neighboring cells. It is found that the character of the system behavior depends essentially on the initial state of the system.

A class of transport models, formulated in terms of finite cells, has been introduced by K. Nagel and was considered by many authors, [4]. The road is represented by one several sequences of cells. Each cell can be occupied by a particle. Particles move at discrete instants in accordance with given rules. The main characteristic of the system is the average velocity of particles. Characteristics of these models were investigated with aid of simulation. M.L. Blank has obtained exact results for deterministic versions of some models of this class, [5].

Exact analytic results for the characteristics of models of this class has been obtained in our papers, [6].

Some one- dimensional and two- dimensional network transport models with a symmetrical periodic structure have been introduced and investigated in [7].

A continuous deterministic dynamical system, which can be also interpreted as transport models, have been introduced and investigated in [8].

Dynamical systems, in which particles move on contours, generating regular one- dimensional or two-dimensional network structures, were studied in [9 – 11].

Movement of particles on a closed one-dimensional periodic structure called a necklace, was considered in [9] and [10]. There are N contours and two, [9], or four, [10], cells on each contour. Each contour has a common cell with each of two neighboring contour. A competition of two particles takes place if these two particles try to to occupy the same common cell. One of competing particles wins the competition with given rules for resolving of the conflict, and the other particle does not move at present time.

A two- dimensional analogue of this system, namely a system with a two- dimensional toroidal system, called a chainmail, was investigated in [11]. In [9–11], the system characteristics have been studied, first at all, the average velocity, defined as the average numbers of transitions of a particle per a time. Conditions of transitions of the system to the state of synergy and collapse were also researched. *Synergy* is defined as the system state such that all particles move after a time instant. *Collapse* is defined as the system state such that all particles do not move after an instant. Interesting intermediate states are observed when the collapse and the synergy appears on a part of the network.

We consider a dynamical system with a discrete state space and discrete time. The system contains N vertices. The vertices send messages to each other. Each vertex sends a message every time instant in accordance with the state of the vertex. The state of each vertex changes in accordance of a given rule at every time instant if no competition takes place. A competition takes place if any two vertices try to send messages to each other at the same instant. The vertex changes its state if there is no competition at present time, or the vertex wins the competition in accordance with a given rule of conflict resolving. The vertex, winning the competition, changes its state, and the the vertex, losing the competition, does not change its state at present time and tries to change its state at next instant.

This dynamical system has been introduced at the plenary lecture on the CMMSE-2014, [11]. The vertex state changes in accordance with a given schedule, Fig. 1.

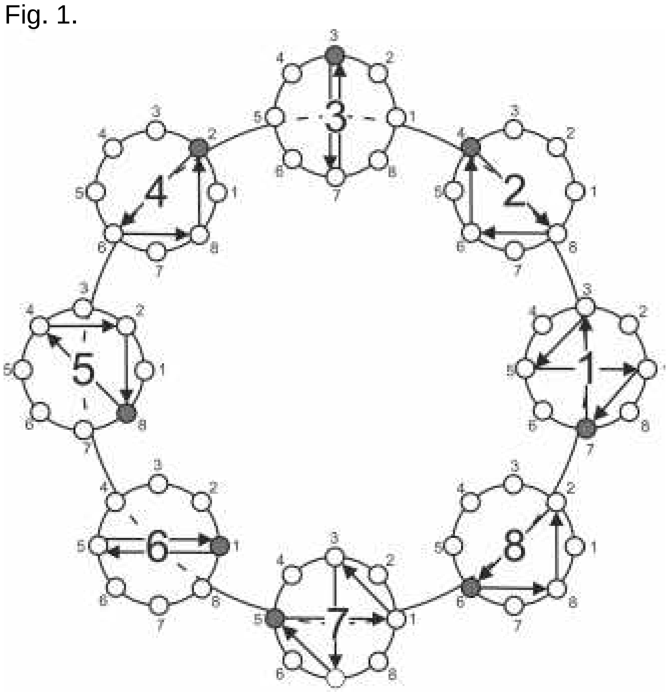

Figure 1: Geometric interpretation in the general case, N=8

We assume that every vertex sends messages successively to each vertex, Fig. 2.

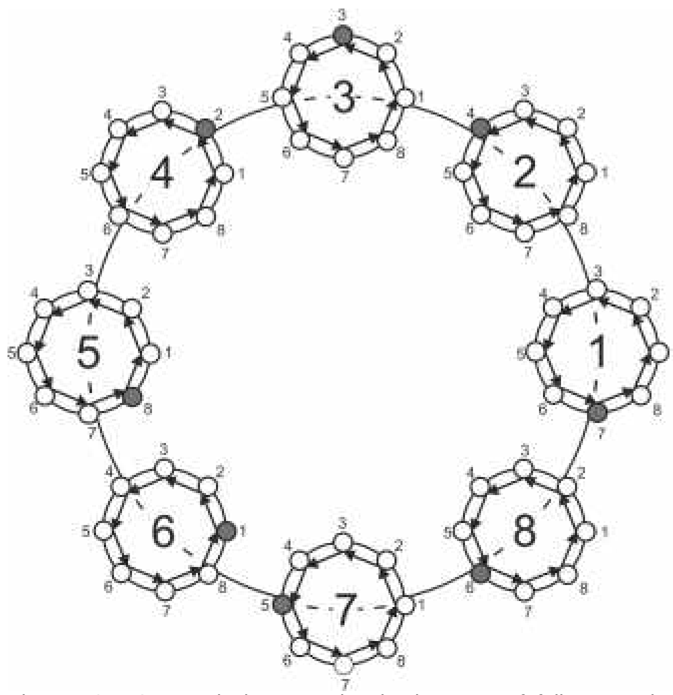

Figure 2: Geometric interpretation in the case of full connection structure, N=8

The vertex productivity is defined as the average number of state changes per a time unit, and the productivity of the system is defined as the average productivity of vertices. The problem is to investigate the productivity of the system and the quantitative properties of the system behavior.

2. FORMULATION OF PROBLEM

2.1. Dynamical system

Consider a dynamical system with discrete time. There are N *vertices* $V_i, \quad i = 1, \ldots, N$. The vertex V_i sends a message to the vertex j if the vertex V_i is in the state $j, \ i, j = 1, \ldots, N$. If appropriate conditions are satisfied, then, the system will be in the state $S_i = j+1$ at next instant. Addition is meant modulo N. Now we describe the conditions of transmission of messages. A *competition* of the vertices V_i and V_j takes place, if, at present time, the vertex V_i sends the message to the vertex V_j, and the vertex V_j sends the message to the vertex V_i, $1 \le i, j \le N$, $i \ne j$. One of competing vertices wins the competition and the other vertex loses the competition. *Suppose each vertex wins the competition with probability 1/2.*

If the vertex V_i is in the state j at time T, and either competes with no vertex or wins all competitions, then the vertex V_i sends a message to the vertex V_i, and the vertex V_i will be, at time $T+1$, in the state $j+1$, addition is meant modulo N. If the vertex loses the competition, this vertex sends no message, and the vertex is still in the state j at time $T+1$.

2.2. Geometric interpretation

Suppose there is an circle and N points on the circle. These points located equidistantly and numbered, e.g., counterclockwise, Fig 1. Each point corresponds to a vertex V_i, of the original system, $1 \le i \le N$. Each point is the center of a small circle. There are N points on each small circle. These points are located similarly to the points of the big circle and are indexed with corresponding numbers. Radius vector, from the point i to the point j of the small circle shows the state S_j of the vertex V_i.

Competitions of the vertices V_i and V_j take place if corresponding radii are collinear and opposite directed.

2.3. System states

The vector

$$S(T) = (S_1, \dots, S_N),$$

where $S_i(T) = j$, is called the *state of the system* at time T, if the vertex V_i is in the state j at time T.

Any state such that each coordinate equals one of the numbers $(1, \dots, N)$ is *permissible,* i.e., any vertex can be in any state, whatever are the states of other vertices. There are N^N permissible states. The system can be represented as a Markov chain [12, 13] with N^N states.

3. CHARACTERISTICS OF SYSTEMS

Denote by $D_i(T)$ the number of state changes of the vertex V_i in the time interval $[0, T)$.

If the limit

$$v_i = \lim_{T \to \infty} \frac{D_i(T)}{T}$$

exists, it is called the *productivity of the vertex* V_i.

The average productivity of vertices

$$v = \frac{1}{N} \sum_{i=1}^{N} v_i,$$

is called the the productivity of the system We say that the system is in the state of synergy after the time T_{syn} if no competitions take place at any time $T \ge T_{syn}$.

If the system comes to the state of synergy, the productivity of any vertex equals 1.

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is called the the *productivity of the system.* We say that the system is in the state of synergy after the time T_{syn} if, beginning from this instant, no competitions take place at any time $T \ge T_{syn}$.

If the system comes to the state of synergy, the productivity of any vertex equals 1 identically.

4. CONDITIONS OF THE SYNERGY

We shall found conditions of synergy and prove that the system reaches the state of synergy after a time interval with a finite expectation.

Theorem 1. *Suppose the system is in the state*

$$S(0) = (S_1(0), \dots, S_N(0))$$

at the initial time instant, and

$$S_i(0) - S_j(0) + i - j = 0, (1)$$

addition is meant modulo N. Then the system is not at the state of synergy.

Proof. Suppose the state $S(0)$ and values i, j satisfy the condition of the theorem. While competitions do not take place, the states of the system at instants $T-1$ and T are connected by the equation

$S_i(T) = S_i(T-1)+1, \quad i=1,\dots,N, \quad T=1,2,\dots,$ addition is meant modulo N, and therefore (1) is still true if $S(0)$ is replaced by $S(0)$. Suppose

$$T^* = j - S_i(0) = i - S_j(0).$$

If conflicts do not take place before the instant T^*, then

$$S_i(T^*) = j,$$

$$S_j(T^*) = i,$$

addition is meant modulo N. Thus a competition of the vertices i and j takes place at time T^*.

Theorem 1 has been proved.

Theorem 2. *There exists a deterministic conflict resolution rule such that the system reaches the state of synergy after a finite time interval.*

Proof. We shall prove that the following rule satisfies the condition of the theorem. The vertices are indexed such that the vertex with lesser number wins the competition.

The proof is by induction on i. Suppose there exists a finite instant t_i such that all vertices with indexes $j=1,\dots,i$ send messages at any instant $T \geq t_i$.

This is true for $i=1$ as, in accordance with the conflict resolution rule, the vertex 1 sends a message at any instant $T \geq t_1 = 0$.

Now we prove that the statement is true for $i = s+1$, if it is true for $i = s$, $s = 1,2,\dots,N-1$.

We still denote by $S_i(T)$ the state of the vertex V_i at time T.

If competitions of the vertex V_{i+1} and a vertex with any lesser number do not take place after the instant t_i, then $t_{i+1} = t_i$, and the statement is true for $s = i+1$.

Suppose that the vertex V_{i+1} and the vertex V_{j^*}, $j^* \leq i$, are competing competes with at time $T_0 \geq t_i$. Then the inequality $i+1-j^*+S_i(0)-S_j(0) \neq 0$ is true for $T > T_0$, and no competitions of the vertices V_{i+1} and V_{j^*} take place. Indeed, the equation $i+1-j^*+S_{i+1}(T)-S_{j^*}(T)=0$, $T > T_0$, were true if the vertex V_{i+1} loses N competitions with vertices with lesser indexes, and no more than one competition of the vertex V_{i+1} and the same other vertex. However there the vertex V_{i+1} can compete with only $N-1$ vertices save the vertex V_{i+1}. This contradiction proves Theorem 2. We suppose below again that each of two competing vertices wins the competition with probability 1/2.

Theorem 3. *The system reaches the state of synergy after a time interval with a finite expectation.*

Proof. Theorem 3 follows from Theorem 1. Indeed, with positive probability, the vertices with lesser indexes will win all competitions until the system comes to the state of synergy.

Let us consider the simplest case. Suppose $N = 2$, $k = 1$. There are 4 states of the system:

$$E_1 = (1,1), E_2 = (1,2), E_3 = (2,1), E_4 = (2,2).$$

Proposition 1. *If $N = 2$, $k = 1$, then the system comes to the state of synergy not more than after 2 steps, and the following sequence of transitions will take place:*

$$E_1 \to E_2 \to E_1 \dots$$

Proof. Proposition 1 is proved by exhaustion.

5. A GENERALIZATION IN THE CASE OF VERTICES, SENDING MESSAGES SIMULTANEOUSLY TO SEVERAL VERTICES

Consider a dynamical system with discrete time. There are N *vertices* V_i, $i=1,\dots,N$. At present time, each vertex sends messages to k successively numbered vertices, $k \leq N$. Each vertex is in one of N states at every time. *If the vertex V_i is in the state $S_i = j$* it attempts to send messages to the vertices $V_{j-k+1}, V_{j-k+2}, \dots, V_j$, addition is meant modulo N. If appropriate conditions are satisfied, then, the system will be in the state $S_i = j+1$ at next instant. Now we describe the conditions of successful transmission of messages. A *competition* of the vertices V_i and V_j takes place, if, at present time, the set of vertices V_{S_i-k+1}, $V_{S_i-k+2},\dots,V_{S_i}$ contains V_j, and the set V_{S_j-k+1}, $V_{S_j-k+2},\dots,V_{S_j}$ contains V_i, $1 \leq i, j \leq N$, $i \neq j$. One of competing vertex wins the competition and the other vertex loses the competition. *Suppose each* *vertex wins the competition with probability 1/2. If a vertex competes with more than one vertex simultaneously, then this vertex wins the competition with each vertex with probability 1/2, and this probability does not depend on the results of other competitions.*

If the vertex V_i is in the state j at time $T,$ and either competes with no vertex or wins all competitions, then the vertex V_i sends messages to all k vertices, and the vertex V_i will be, at time $T+1,$ in the state $j+1,$ addition is meant modulo $N.$ If the vertex lose at least one competition, this vertex sends no message, and the vertex still in the state j at time $T+1.$

We remark that, in whatever state is the system at T the system will be in another state at time $T+1$ with positive probability. Indeed, one of the competing vertices wins all competitions with positive probability. Hence, with positive probability, the state of this vertex changes, and therefore the state of the system also changes.

Theorem 4. *Suppose the system is in the state*

$$S(0)=(S_1(0),\ldots,S_N(0))$$

at the initial time instant, and there exist integer numbers $i,\ j,\ d_i,\ d_j,\ 0\le d_i,d_j<m,\ i\ne j,$ *such that*

$$S_i(0)-S_j(0)+i-j+d_i-d_j=0,(1)$$

addition is meant modulo $N.$ *Then the system is not at the state of synergy.*

Proof. The proof of Theorem 4 is similar to the proof of Theorem 1.

Theorem 5. *Suppose* $k\ge 2.$ *Then the system cannot be in the state of synergy.*

Proof. if the system, being in the state $(S_1,\ldots,S_N),$ is in the state of synergy, then the inequality

$$S_i-S_j+i-j+d_i-d_j\ne 0$$

is true for any $i,\ j,\ d_i,\ d_j,\ 0\le d_i,d_j<m,\ i\ne j,$ $1\le i,j\le N,$ addition is meant modulo $N,$ for any $i,$ $j,\ d_i,\ d_j,\ 0\le d_i,d_j<m,\ i\ne j,\ 1\le i,j\le N.$ Therefore values of $S_i+i+d_i,\ 0\le d_i<m,$ $i=1,\ldots,N,$ must be different modulo $N.$ However it is impossible that these mN values are different modulo $N.$

Theorem 5 has been proved. The following theorem gives an upper estimation of the productivity of the system.

Theorem 6. *Suppose* $k\ge 2.$ *Then*

$$v\le 1-\frac{1}{N^2},$$

where v *is the productivity of the system.*

Proof. Taking into account Theorem 3, we see that, for any state $S=(S_1,\ldots,S_N),$ there are exist integer numbers $i,\ j,\ d_i,\ d_j,\ 0\le d_i,d_j<m,\ i\ne j,$ such that the equation

$$S_i-S_j+i-j+d_i-d_j=0.$$

is true. A competition of the vertices V_i and V_j take place no more than after $N-1$ steps. Therefore at least one competition takes place at each N steps, and one of the competing vertices does not move. From this, Theorem 6 follows. Let us consider an example.

Suppose $N=3,\ k=2.$ Consider the behavior of the system in this case. Let us consider a Markov chain corresponding to the system. There are 27 states of the system:

$$E_l=(i,j,k),$$

$$l=9(i-1)+3(j-1)+k, 1\le i,j,k\le 3.$$

Proposition 2. *Suppose* $N=3,\ k=2.$ *Then it is true the following:*

(1) All states of the system except the state E_6 *form a single state of aperiodic communicating states. The system cannot be at the state* E_6 *at any time* T *except* $T=0.$

(2) The productivity of the system equals 0.5428.

Proof. The first statement is proved by exhaustion.

Since 26 states form a single class of aperiodic communicating states, there positive steady probabilities of these states.

Denote by p_i the steady probability of the state $E_i.$ The system of equations for the steady probabilities is satisfied by the following values:

$$p_1=0.03828, p_2=0.05741, p_3=0.04785, p_4=0.02871,$$

$$p_5=0.04785, p_7=0.01914, p_8=0.01914, p_9=0.02871,$$

$$p_{10} = 0.03828, p_{11} = 0.01914, p_{12} = 0.01914, p_{13} = 0.01914,$$
$$p_{14} = 0.03828, p_{15} = 0.02871, p_{16} = 0.04306, p_{17} = 0.03828,$$
$$p_{18} = 0.01914, p_{19} = 0.01914, p_{20} = 0.15311, p_{21} = 0.05741,$$
$$p_{22} = 0.01914, p_{23} = 0.05741, p_{24} = 0.04785, p_{25} = 0.03828,$$
$$p_{26} = 0{,}01914,\ p_{27} = 0.03828.$$

The value of the vertex i productivity can be calculated as

$$v_i = \sum_{j \neq 6} p_j a_{ji},$$

where a_{ij} is the probability of the vertex i changes its state at present time if the system is in the state E_j. We have

$$v_1 = v_2 = v_3 = v = 0.5428.$$

Proposition 2 has been proved. Suppose the state S_6 is initial, Fig. 3.

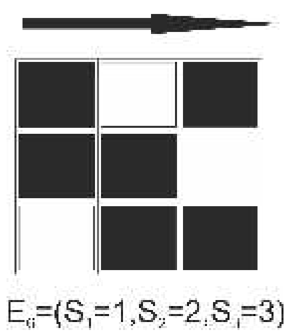

Figure 3: Inessential state E_6

The behavior of the system is shown in Fig. 4.

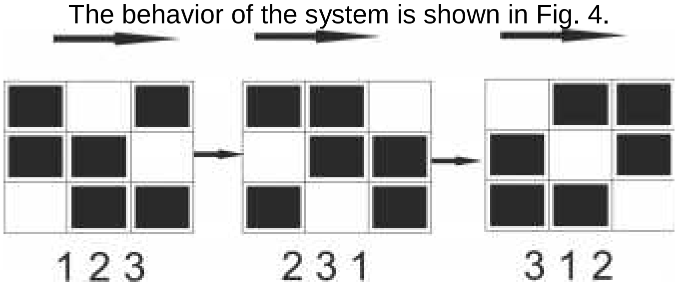

Figure 4: Behavior of the system in the case the state E_6 is initial

6. CONCLUSION

We have introduced and investigated a dynamical system. The concepts of the system productivity and the synergy have been introduced and investigated.

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